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On the classical dynamics of charges in non-commutative QED

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Abstract. Following Wong's approach to formulating the classical dynamics of charged particles in non-Abelian gauge theories, we derive the classical equations of motion of a charged particle in U(1) gauge theory on non-commutative space, the so-called non-commutative QED. In the present use of the procedure, it is observed that the definition of the mechanical momenta should be modified. The derived equations of motion manifest the previous statement about the dipole behavior of the charges in non-commutative space.

In the last years much attention has been paid to the formulation and study of field theories on non-commutative spaces. Apart from the abstract mathematical interests, there are various physical motivations for doing so. One of the original motivations has been to get "finite" field theories via the intrinsic regularizations which are encoded in some non-commutative spaces [1]. The other motivation comes from the unification aspects of theories on non-commutative spaces. These unification aspects have been the result of the "algebraization" of "space, geometry and their symmetries" via the approach of non-commutative geometry [2]. Interpreting the Higgs fields of the theories with spontaneously broken symmetries as gauge fields in the discrete directions of multi-sheet spaces is an example of this point of view on non-commutative spaces [3]. The other motivation refers to the natural appearance of non-commutative spaces in some areas of physics and recently in string theory. It has been understood that string theory is involved by some kinds of non-commutativities; two examples are
(1) the coordinates of bound states of N D-branes are represented by $N \times N$ Hermitian matrices [4], and
(2) the longitudinal directions of D-branes in the presence of a B-field background appear to be non-commutative, as seen by the ends of open strings [5–7]. In the latter case, we encounter a spacetime in which the coordinates satisfy the canonical commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{1}$$

in which $\theta^{\mu\nu}$ is a constant second rank tensor. Since the coordinates do not commute, any definition of functions or fields should be performed under a prescription for ordering of coordinates, and a natural choice is the symmetric one, the so-called Weyl ordering. To any function $f(x)$ on ordinary space, one can assign an operator $\hat{O}_f$ by

$$\hat{O}_f(\hat{x}) := \frac{1}{(2\pi)^n} \int \mathrm{d}^n k \; \tilde{f}(k)\, \mathrm{e}^{-\mathrm{i}k\cdot\hat{x}}, \tag{2}$$

in which $\tilde{f}(k)$ is the Fourier transform of $f(x)$ defined by

$$\tilde{f}(k) = \int \mathrm{d}^n x \; f(x)\, \mathrm{e}^{\mathrm{i}k\cdot\, x}. \tag{3}$$

Due to the presence of the phase $\mathrm{e}^{-\mathrm{i}k\cdot\hat{x}}$ in the definition of $\hat{O}_f$, we recover the Weyl prescription for the coordinates. In a reverse way we also can assign to any symmetrized operator a function or field living on the non-commutative plane. Also, we can assign to the product of any two operators $\hat{O}_f$ and $\hat{O}_g$ another operator as follows:

$$\hat{O}_f \cdot \hat{O}_g =: \hat{O}_{f\star\, g}, \tag{4}$$

in which f and g are multiplied under the so-called $\star$-product defined by

$$(f \star g)(x) = \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} f(x)g(y)\,|_{y=x}\,. \tag{5}$$

By all this one learns how to define physical theories in non-commutative spacetime, and eventually it appears that the non-commutative field theories are defined by actions that are essentially the same as in ordinary spacetime, with the exception that the products between fields are replaced by $\star$-products; see [8] for a review. Though the $\star$-product itself is not commutative (i.e., $f \star g \neq g \star f$) the following identities make some of calculations easier in field theories:

$$\int (f \star g)(x) \mathrm{d}^n x = \int (g \star f) \mathrm{d}^n x$$

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$$= \int (f \cdot g)(x) \mathrm{d}^n(x), \tag{6}$$

$$\int (f \star g \star h)(x) \mathrm{d}^n x = \int (f \cdot (g \star h))(x) \mathrm{d}^n x$$
$$= \int ((f \star g) \cdot h)(x) \mathrm{d}^n x, \tag{7}$$

$$\int (f \star g \star h)(x) \mathrm{d}^n x = \int (h \star f \star g)(x) \mathrm{d}^n x$$
$$= \int (g \star h \star f)(x) \mathrm{d}^n x. \tag{8}$$

By the first two ones we see that in the integrands always one of the $\star$'s can be removed.

Non-commutative QED (NCQED) is given by the action

$$S = \int \mathrm{d}^4 x \tag{9}$$
$$\times \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \overline{\psi} \gamma^\mu (\partial_\mu - \mathrm{i} g A_\mu \star) \psi - \frac{\mathrm{i} m c}{\hbar} \overline{\psi} \psi \right),$$

in which the field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} g [A_\mu, A_\nu]_{\mathrm{MB}}, \tag{10}$$

in which the commutator is defined by Moyal by

$$[A_\mu, A_\nu]_{\mathrm{MB}} = A_\mu \star A_\nu - A_\nu \star A_\mu. \tag{11}$$

The action is invariant under the transformations

$$\psi \longrightarrow \psi' = U \star \psi,$$
$$\overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi} \star U^{-1},$$
$$A_\mu \longrightarrow A'_\mu = U \star A_\mu \star U^{-1} + \frac{\mathrm{i}}{g}\, U \star \partial_\mu U^{-1}, \tag{12}$$

in which $U(x)$ is the $\star$-phase ($U \star U^{-1} = U^{-1} \star U = 1$) defined by a function $\lambda(x)$ via the $\star$-exponential:

$$U(x) = \exp_\star(\mathrm{i}\lambda) = 1 + \mathrm{i}\lambda - \frac{1}{2} \lambda \star \lambda + \ldots \tag{13}$$

Under the gauge transformation, the field strength transforms as

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = U \star F_{\mu\nu} \star U^{-1}. \tag{14}$$

We mention that the transformations of the gauge fields as well as the field strength look like those of non-Abelian gauge theories. Besides, we see that the pure gauge field sector of the action contains terms which are responsible for interaction between the gauge particles, again as in the situation we have in non-Abelian gauge theories.

Among others, there is one approach due to Wong [9] for the derivation of the classical equations of motion of particles that have non-Abelian charges. In this formulation there are a couple of equations among which one is for the dynamics of the charged particle in spacetime, and one for the dynamics of the isospin charge of the particle, as an internal degree of freedom. The former is analogous to the Lorentz force in electro-magnetism. Noting the non-Abelian nature of NCQED, it is quite reasonable to use the approach by Wong to derive the classical equations of motion for charges in NCQED.

Let us review briefly Wong's approach in the next lines; see [10][1]. The equations of motion for the fermionic matter field in the fundamental representation and in the presence of a background field $A_\mu(x)$ is

$$\gamma^\mu (\partial_\mu - \mathrm{i} g A^a_\mu \hat{T}_a) \Psi(x) + \frac{\mathrm{i} m c}{\hbar} \Psi(x) = 0, \tag{15}$$

in which $\hat{T}_a$'s are the generators of the group, satisfying $[\hat{T}_a, \hat{T}_b] = \mathrm{i} f^c_{ab} \hat{T}_c$. Viewing this equation as a Schrödinger equation (recalling $\partial_0 = c^{-1} \partial_t$) one reads the Hamiltonian as

$$\hat{H} = c\alpha^i (\hat{p}_i - g\hbar A^a_i(\hat{x}) \hat{T}_a) + mc^2 \beta - gc\hbar A^a_0(\hat{x}) \hat{T}_a, \tag{16}$$

in which α^i and β are the Dirac matrices, and $\hat{p}_i$ is for $-\mathrm{i}\hbar\partial_i$. In the Heisenberg picture, one obtains the equations of motion for operators:

$$\dot{\hat{x}}^i = \frac{\mathrm{i}}{\hbar} [\hat{H}, \hat{x}^i] = c\alpha_i, \tag{17}$$

$$\dot{\hat{p}}_i = \frac{\mathrm{i}}{\hbar} [\hat{H}, \hat{p}_i] = gc\hbar (\alpha^j \partial_i A^a_j + \partial_i A^a_0) \hat{T}_a, \tag{18}$$

$$\dot{\hat{T}}_a = \frac{\mathrm{i}}{\hbar} [\hat{H}, \hat{T}_a] = -g f^c_{ab} (\dot{\hat{x}}_i A^b_i + c A^b_0) \hat{T}_c. \tag{19}$$

By defining the mechanical momenta by $\hat{\pi}_i := \hat{p}_i - g\hbar A_i(\hat{x})$, one gets

$$\dot{\hat{\pi}}_i = g\hbar (c F^a_{i0} + \dot{\hat{x}}^j F^a_{ij}) \hat{T}_a, \tag{20}$$

in which the $F_{\mu\nu}$'s are the field strengths of the non-Abelian gauge theory, defined by $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^a_{bc} A^b_\mu A^c_\nu$. By comparison of this equation with that of electro-magnetism Wong suggests the following for the non-Abelian case:

$$m\ddot{\xi}_\mu = g(F^a_{\mu\nu} T_a) \dot{\xi}^\nu, \tag{21}$$

in which $\xi^\mu(\tau)$ represents the world-line of the particle, and we used the change $\hbar\hat{T}_a \to T_a$. The dot in the equation is for a derivative with respect to the proper-time. We mention that in the above equation the T_a cannot be and are not supposed to be operators (i.e., matrices) anymore, while we interpret them as number functions capturing the degrees of freedom coming from the group structure, satisfying the equations of motion:

$$\dot{T}_a + g\dot{\xi}^\mu f^c_{ab} A^b_\mu T_{\mathrm{c}} = 0. \tag{22}$$

From this we learn that the group degrees of freedom, also known as isotopic spin, perform a precessional motion: $\mathrm{d}/\mathrm{d}\tau (T_a T^a) = 0$.

[1] Note the missing $\mathrm{i} = \sqrt{-1}$ before of the mass term in [10].

Now we use Wong's method for the case of NCQED, and we do this in the first order of the non-commutativity parameter $\theta_{\mu\nu}$. The Lagrangian in this order is

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \overline{\psi}\gamma^{\mu}(\partial_{\mu} - \mathrm{i}gA_{\mu})\psi - \frac{1}{2}g\overline{\psi}\gamma^{\mu}\theta^{\alpha\beta}\partial_{\alpha}A_{\mu}\partial_{\beta}\psi - \frac{\mathrm{i}mc}{\hbar}\overline{\psi}\psi + O(\theta^2), \quad (23)$$

in which the field strength is

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + g\theta^{\alpha\beta}\partial_{\alpha}A_{\mu}\partial_{\beta}A_{\nu} + O(\theta^2). \quad (24)$$

The action corresponding to the Lagrangian (23) is invariant under the first-order transformations in θ[2]:

$$\psi \longrightarrow \psi' = \mathrm{e}^{\mathrm{i}\lambda}\left(\psi - \frac{\theta^{\mu\nu}}{2}\partial_{\mu}\lambda\partial_{\nu}\psi\right) + O(\theta^2),$$

$$\overline{\psi} \longrightarrow \overline{\psi}' = \mathrm{e}^{-\mathrm{i}\lambda}\left(\overline{\psi} - \frac{\theta^{\mu\nu}}{2}\partial_{\mu}\lambda\partial_{\nu}\overline{\psi}\right) + O(\theta^2),$$

$$A_{\mu} \longrightarrow A'_{\mu} = A_{\mu} + \frac{1}{g}\partial_{\mu}\lambda - \theta^{\alpha\beta}\partial_{\alpha}\lambda\,\partial_{\beta}A_{\mu} - \frac{\theta^{\alpha\beta}}{2g}\partial_{\alpha}\lambda\,\partial_{\beta}\partial_{\mu}\lambda + O(\theta^2). \quad (25)$$

The equation of motion for ψ is obtained to be

$$\gamma^0(\partial_0 - \mathrm{i}gA_0)\psi + \gamma^i(\partial_i - \mathrm{i}gA_i)\psi + \frac{1}{2}g\gamma^{\mu}\theta^{\alpha\beta}\partial_{\alpha}A_{\mu}\partial_{\beta}\psi + \frac{\mathrm{i}mc}{\hbar}\psi = 0. \quad (26)$$

Hereafter we assume that non-commutativity is just for the spatial directions: $\theta^{0\mu} = \theta^{\mu 0} = 0$. So, the above equation appears in the form:

$$\gamma^0(\partial_0 - \mathrm{i}gA_0)\psi + \gamma^i(\partial_i - \mathrm{i}gA_i)\psi + \frac{1}{2}g\theta^{ij}\gamma^{\mu}\partial_i A_{\mu}\partial_j\psi + \frac{\mathrm{i}mc}{\hbar}\psi = 0. \quad (27)$$

Again viewing this as a Schrödinger equation we read the corresponding Hamiltonian as

$$\hat{H} = -gc\hbar A_0 + c\alpha^i(\hat{p}_i - g\hbar A_i) + \frac{1}{2}gc\theta^{ij}\alpha^{\mu}\partial_i A_{\mu}\hat{p}_j + mc^2\beta, \quad (28)$$

in which we have used $\alpha^{\mu} = (\alpha^0, \alpha^k) = (I, \alpha^k)$. The Heisenberg equations of motion are derived for the operators as well:

$$\dot{\hat{x}}^l = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{x}^l] = c\alpha^l + \frac{1}{2}gc\theta^{il}\alpha^{\mu}\partial_i A_{\mu}, \quad (29)$$

$$\dot{\hat{p}}_l = \frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{p}_l] = gc\hbar\alpha^{\mu}\partial_l A_{\mu} - \frac{1}{2}gc\theta^{ij}\alpha^{\mu}\partial_l\partial_i A_{\mu}\hat{p}_j. \quad (30)$$

From the first equation we have $c\alpha^l = \dot{\hat{x}}^l - \frac{1}{2}gc\theta^{il}\partial_i A_0 - \frac{1}{2}g\theta^{il}\dot{\hat{x}}^k\partial_i A_k + O(\theta^2)$; this will be used for later replacements. For the case of NCQED, we see that the interaction between fermions and gauge fields is different in comparison with ordinary (Abelian and non-Abelian) gauge theories. For the present case we have the following mechanical momenta:

$$\hat{\pi}_l = \hat{p}_l - g\hbar A_l(\hat{x}) + \frac{1}{2}g\theta^{ij}\partial_i A_l(\hat{x})\hat{p}_j. \quad (31)$$

This form of the mechanical momenta can be read also from the covariant derivative of NCQED, $D_{\mu}\psi = \partial_{\mu}\psi - \mathrm{i}gA_{\mu}\star\psi$, which changes by a similarity transformation under gauge transformations. After this modification to the Wong's approach, one can calculate the time derivative as

$$\dot{\hat{\pi}}_l = \dot{\hat{p}}_l - g\hbar\partial_t A_l - \mathrm{i}g[\hat{H}, A_l] + \frac{1}{2}g\theta^{ij}\partial_t\partial_i A_l\hat{p}_j + \frac{1}{2}g\theta^{ij}\frac{\mathrm{i}}{\hbar}[\hat{H}, \partial_i A_l\hat{p}_j]. \quad (32)$$

After sufficient manipulations and replacements, and omitting hats we obtain

$$\begin{aligned}\dot{\pi}_l = {} & gc\hbar(\partial_l A_0 - \partial_0 A_l) + g\hbar\dot{x}^i(\partial_l A_i - \partial_i A_l) \\ & + \frac{1}{2}gc\theta^{ij}p_j(\partial_0\partial_i A_l - \partial_l\partial_i A_0) \\ & + \frac{1}{2}g\theta^{ij}\dot{x}^k p_j(\partial_k\partial_i A_l - \partial_l\partial_i A_k) \\ & - \frac{1}{2}g^2c\hbar\theta^{ij}\left(\partial_i A_0 + \frac{1}{c}\dot{x}^k\partial_i A_k\right)(\partial_j A_l + \partial_l A_j) \\ & + O(\theta^2).\end{aligned} \quad (33)$$

By defining the field strengths:

$$f_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}, \quad (34)$$

$$F_{\mu\nu} = f_{\mu\nu} + g\theta^{ij}\partial_i A_{\mu}\partial_j A_{\nu}, \quad (35)$$

and by setting $g\hbar c = e$ and $\mu^i := \frac{1}{2}gc\theta^{ij}p_j = \frac{e}{2\hbar}\theta^{ij}p_j$, we get

$$\dot{\pi}_l = e\left(F_{l0} + \frac{1}{c}\dot{x}^i F_{li}\right) + \mu^i\partial_i f_{0l} + \frac{1}{c}\mu^i\dot{x}^k\partial_i f_{kl} + \frac{1}{c}\dot{\mu}^i f_{li} + O(\theta^2). \quad (36)$$

The first two terms are easily understood as the dynamics of a charged particle in the background of the non-commutative field strength $F_{\mu\nu}$. To understand the other terms, we compare the result with those of a dipole electric in the background of ordinary electro-magnetic fields. The corresponding Lagrangian for a point-like electric dipole can easily be derived by considering the dynamics of two

[2] The first-order transformations can be obtained by noting the fact that the $\star$-power of a function $f(x)$ behaves as follows: $f^n_{\star} := f\star f\star\ldots\star f = f^n + O(\theta^2)$, and hence we have $U(x) = \exp_{\star}(\mathrm{i}\lambda) = \mathrm{e}^{\mathrm{i}\lambda} + O(\theta^2)$.

equal mass particles with opposite charges q and $-q$, while their relative distance ℓ is small, defining the electric dipole $\mu := q\ell$. So the starting point is

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{1}{c}\mu^i\dot{x}^i f_{ji} + \mu^i f_{i0}. \quad (37)$$

So the equation of motion for the position of the dipole appears as

$$m\ddot{x}_k = \mu^j\partial_k f_{0j} - \frac{1}{c}\dot{\mu}^j f_{jk} + \frac{1}{c}\mu^j\dot{x}^i(\partial_k f_{ji} - \partial_i f_{jk}). \quad (38)$$

After using the Maxwell equation of ordinary electro-magnetism[3], one ends up with the equation like that for charges in NCQED.

The result here by Wong's approach in an arbitrary background field has been pointed out also via the behavior of open strings ending on D-branes in the presence of a B-field [11], and also were obtained through the study of the implications of possible non-commutativity in the present world in some specific examples [12, 13]. From the string theory point of view the situation can be described as below. For example, the mode expansion of open string coordinates ending on a D2-brane is given by [11]

$$\begin{aligned} X^0 &= x^0 + p^0\tau + \sum_{n\neq 0} a_n^0 \frac{\mathrm{e}^{-in\tau}}{n}\cos n\sigma, \\ X^i &= x^i + (p^i\tau - B^i_j p^j\sigma) \\ &\quad + \sum_{n\neq 0}\frac{\mathrm{e}^{-in\tau}}{n}(\mathrm{i}a_n^i\cos n\sigma + B^i_j a^j \sin n\sigma), \quad i = 1, 2, \\ X^b &= x^b + p^b\tau + \sum_{n\neq 0} a_n^b \frac{\mathrm{e}^{-in\tau}}{n}\cos n\sigma, \quad b = 3, \ldots, 9, \end{aligned} \quad (39)$$

in which the B^i_j are components of the B-field background. Now we see that even for the case in which the oscillations are suppressed, the distance between the ends of open strings on the D2-brane is not zero, appearing to be $\Delta^i = X^i(\sigma = 0, \tau) - X^i(\sigma = \pi, \tau) = \pi B^i_j p^j$, by which we expect a dipole behavior due to the $\pm$ charges we assign to the ends of oriented open strings [4, 11]. This behavior of open strings has been suggestive in formulating a theory for fields of dipoles rather than for fields with the property that their quanta are particles [14]. The rule of multiplication of fields for dipoles is reminiscent of the $\star$-product.

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[3] And using the identity $\epsilon_{jil}\partial_k - \epsilon_{jkl}\partial_i = \epsilon_{kil}\partial_j - \epsilon_{kij}\partial_l$, for $i, j, k, l = 1, 2, 3$.